The Sensitivity Conjecture

$f : \{0,1\}^n \to \{0,1\}$ $\quad x \in \{0,1\}^n$ $\quad i \in [n]$ $\quad B \subset [n]$

$x^{(i)} = x_1, \dots, 1-x_i, \dots, x_n$

Coordinate i is sensitive (for x) if $f(x) \neq f(x^{(i)})$

$$S_x(f) = \left|\{i \in [n] : i \text{ is sensitive}\}\right|$$

$$s(f) = \max_x S_x(f)$$

$$\left(x^{(B)}\right)_i = \begin{cases} 1-x_i & i \in B \\ x_i & i \notin B \end{cases}$$

B is a sensitive block (for x) if $f(x) \neq f(x^{(B)})$

$$bs_x(f) = \max\left\{a : \begin{array}{l}\text{exist } B_1, \dots, B_a \\ \text{pairwise disjoint} \\ \text{sensitive blocks}\end{array}\right\}$$

$$bs(f) = \max_x bs_x(f)$$

By definition, $s(f) \leq bs(f)$ (consider blocks of size 1)

The Sensitivity Conjcture: there exists C, s.t. $bs(f) = O(s(f)^C)$

Basic Examples of boolean functions

$x \in \{0,1\}^n$

$|x|$ is the number of 1's.

$AND_n(x) = 1$ if $|x| = n$ $\quad s = bs = n$

$OR_n(x) = 1$ if $|x| \geq 1$ $\quad s = bs = n$

$MAJ_n(x) = 1$ if $|x| \geq \frac{n}{2}$ $\quad s = bs = \lceil \frac{n}{2} \rceil$

$PAR_n(x) = 1$ if $|x|$ is odd $\quad s = bs = n$

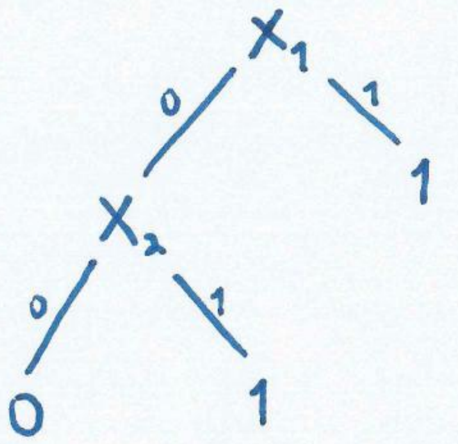
More Complexity Measures

A certificate is an assignment to a subset of $\{x_1, \dots, x_n\}$, that defines uniquely the value of $f(x)$.

$C_x(f)$ = size of minimal certificate agreeing with x.

$C(f) = \max_x C_x(f)$

A decision tree describes an algorithm for evaluating f. At each step, one bit of x is read. The depth is the number of bits read in the worst case.



$f = ?$

$D(f)$ is the minimal possible depth of a decision tree for f.

$C(f) \leq D(f)$ since once the algorithm terminates, the read bits form a certificate.

Gaps

$s \leq bs \leq C \leq D$

$bs_x(f) \leq C_x(f)$ since blocks are disjoint and each one intersects the certificate.

$D \leq C^2$

$C \leq s \cdot bs \leq bs^2$

The Degree of a Boolean Function

Claim: Every fuction $f:\{0,1\}^n \to \mathbb{R}$ is uniquely represented as a multilinear real polynomial.

proof: for fixed $y \in \{0,1\}^n$, $\prod_{i\in[n]} \left(\begin{cases} 1-x_i & y_i=0 \\ x_i & y_i=1 \end{cases} \right) = f(x) := \begin{cases} 1 & x=y \\ 0 & x\neq y \end{cases}$

$$\dim \left\{ \begin{matrix} \text{multilinear polynomials} \\ \text{of } x_1,\dots,x_n \end{matrix} \right\} = 2^n = \dim \left\{ \begin{matrix} \text{functions} \\ f:\{0,1\}^n \to \mathbb{R} \end{matrix} \right\}$$

$\blacksquare$

$\deg(f)$ is the degree of f as a multilinear polynomial.

Proof of the sensitivity conjcture:

$\deg(f) \geq \sqrt{\frac{bs(f)}{2}}$ Nisan, Szegedy (1992)

$S(f) \geq \sqrt{\deg f}$ Hao Huang (2019)
using equivalence by Gotsman and Linial (1992)

Equivalence by Gotsman and Linial

Change convention: $f:\{-1,+1\}^n \to \{-1,+1\}$. $\deg(f)$ remains the same.

Question: given $\deg f$, how low can $s(f)$ be?

WLOG assume $\deg(f) = K$ and $x_1 \cdot \ldots \cdot x_K$ a maxinomial.

substitute $x_{K+1} = \ldots = x_n = 1$.

$\deg(f)$ is unchanged, $s(f)$ not increased $\Rightarrow$ WLOG, assume $\deg(f) = n$

Take $g(x_1, \ldots, x_n) := x_1 \cdot \ldots \cdot x_n \cdot f(x_1, \ldots, x_n)$

$$\deg(f) = n \iff g(0,\ldots,0) \neq 0 \iff \sum_{x \in \{-1,+1\}^n} g(x) \neq 0$$

Q_n is the hypercube graph with vertices $\{-1,+1\}^n$.

Let $S := \{x \in \{-1,+1\}^n : g(x) = 1\}$. Let $G := Q_n[S] \cup Q_n[S^c]$.

$s_x(f) = n - s_x(g) = d_G(x)$

$s(f) = \Delta(G)$

Question: $|S| \neq 2^{n-1}$ how low can $\Delta(Q_n[S] \cup Q_n[S^c])$ be?

Q_n is the hypercube graph. $V(Q_n) = \{0,1\}^n$

Theorem (Hao Huang, 2019):

Assume $S \subset \{0,1\}^n$ and $2^{n-1}+1 \leqslant |S|$. Then $\Delta(Q_n[S]) \geq \sqrt{n}$

proof:

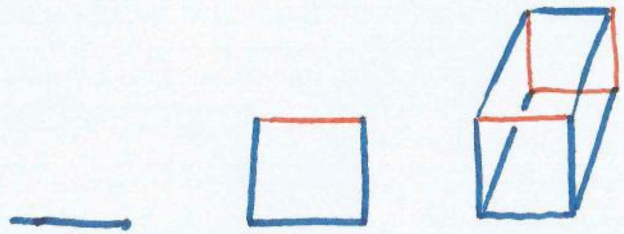
$A_0 = [0]$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_2 = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array} \quad \cdots$$

$$A_n = \begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{bmatrix}$$

Identify $\{0,1\}^n$ with $\{0,\dots,2^n-1\}$ by binary representation.

Then: $$(A_n)_{xy} = \begin{cases} \pm 1 & xy \in E(Q_n) \\ 0 & \text{o/w} \end{cases}$$



A_1 A_2 A_3

$(A_n)^2 = n I_{2^n}$ calculation: $\begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}$

$$\begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix} \begin{bmatrix} (A_{n-1})^2+I & A_{n-1}-A_{n-1} \\ A_{n-1}-A_{n-1} & (A_{n-1})^2+I \end{bmatrix}$$

so by induction, $(A_n)^2 = \begin{bmatrix} (n-1)I+I & 0 \\ 0 & (n-1)I+I \end{bmatrix}$

All eigenvalues are $\pm\sqrt{n}$. $tr(A_n)=0$. A_n is symetric.

Hence A_n has diagonal form: $\begin{bmatrix} \sqrt{n} I_{2^{n-1}} & 0 \\ 0 & -\sqrt{n} I_{2^{n-1}} \end{bmatrix}$.

Let $U = \left\{ v \in \mathbb{R}^{2^n} : \begin{array}{l} v_x = 0 \text{ for} \\ \text{all } x \notin S \end{array} \right\}$

$\dim U \geq 2^{n-1}+1$, eigenspace of $\sqrt{n}$ has dim. 2^{n-1}.

so let $v \in U$ with $A_n v = \sqrt{n}\, v$ and $v \neq 0$.

Let x maximize $|v_x|$. (so $x \in S$)

$$|(A_n v)_x| = \left| \sum_{y \in \{0,1\}^n} (A_n)_{xy} v_y \right| \leq \sum_{y \in \{0,1\}^n} |(A_n)_{xy}| \cdot |v_y| = \sum_{\substack{y \in S \\ xy \in E(Q_n)}} |v_y| \leq d_{Q_n(S)}(x) \cdot |v_x|$$

$$|(A_n v)_x| = |\sqrt{n}\, v_x| = \sqrt{n}\, |v_x|$$

Conclusion:

$\sqrt{n} \leq d_{Q_n(S)}(x) \leq \Delta Q_n(S)$

Cauchy's Interlace Theorem

Let A be an $n \times n$ matix and let B be an $m \times m$ principal submatrix.

Let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of A
and $\mu_1 \geq \dots \geq \mu_m$ the eigenvalues of B.

Then for $1 \leq i \leq m$, $\lambda_i \geq \mu_i \geq \lambda_{i+n-m}$.
In other words, the i'th largest eigenvalue of B is at most the i'th largest eigenvector of A, and the i'th smallest eigenvector of B is at least the i'th smallest eigenvector of A.

Sketch of proof for $\deg(f) \geq \sqrt{\frac{bs(f)}{2}}$ by Nisan and Szegedy

Consider the symetrization of the polynomial f.

$$f^{sym}(x_1 \dots x_n) = \frac{1}{n!} \sum_{\pi \in S_n} f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

f^{sym} is a sum of elementary symmetric polynomials:

$$S_k = \sum_{\substack{A \subseteq [n] \\ |A| = k}} \prod_{i \in A} x_i \underset{\substack{\uparrow \\ \text{for} \\ x \text{ in } \{0,1\}^n}}{=} \binom{x_1 + \dots + x_n}{k}$$

denote $z = x_1 + \dots + x_n$.

$$S_k(x_1 + \dots + x_n) = p_k(z) = \frac{1}{k!} z(z-1) \dots (z-k+1)$$

so there is p, with $p(z) = f^{sym}(x)$ and $\deg(p) \leq \deg(f)$

Assume $f(x) = 0$ for $z = 0$
and $f(x) = 1$ for $z = 1$

$p(0)=0$, $p(1)=1$, $0 \le p(z) \le 1$ for ~~integer~~ $z = 0, 1, \dots, n$

By a result from Aproximation Theory (The Markov brothers' inequality)

conclude $\deg(p) \ge \sqrt{\frac{n}{2}}$.

for general f with $bs(f) = k$. Assume WLOG $x = (0, \dots, 0)$, $f(x) = 0$, and $B_1, \dots, B_k$ disjoint sensitive blocks.

define $g(x_1, \dots, x_k) = f(y)$ with ~~...~~ $y_i = x_j$ if $i \in B_j$ for some j, and 0 otherwise.

Example : "AND of ORs"

Let $n = k^2$. f accepts a boolean $k \times k$ matrix x.

A row of x is "good" if it has at least one 1.

$f(x)$ is 1 if ~~x has at least one good row.~~ all the rows of x are good.

$$f(x) = \underset{i \in [k]}{\mathrm{AND}}\left(\underset{j \in [k]}{\mathrm{OR}} (x_{ij}) \right)$$

$$s(f) = \sqrt{n}$$

$$f(x) = \prod_{i \in [k]} \left(1 - \prod_{j \in [k]} (1 - x_{ij}) \right) \Rightarrow \deg(f) = n$$

$$\boxed{s(f) = \sqrt{\deg(f)}}$$

Example: "Adress function"

$n = k + 2^k$.

Interpret $x_1 \dots x_k$ as a number between 0 and $2^k - 1$.

Define $f(x) = x_{k+1+x_1 \dots x_k}$

example:

$$f(\overbrace{10}^{2}0110) = 1$$

$$s(f) = k+1 \sim \log n$$

$$D(f) = k+1$$

Example "Pairs of 1's"

Let $n = 4k^2$. f accepts a $2k \times 2k$ boolean matrix, X.

A row of X is "good" if it has exactly two 1's, and they are consecutive.

$f(X) = 1$ if X has a good row.

$s(f) = 8k = 2\sqrt{n}$

$bs(f) = \frac{n}{2}$

for X the all-0 matrix, consider the set of sensitive blocks:

$$\left\{ \{(i,j),(i,j+1)\} \;\middle|\; \begin{matrix} i \in [2k] \\ j \in 2[k]-1 \end{matrix} \right\}$$