Random high-density packings of 2×2 tiles on the square lattice

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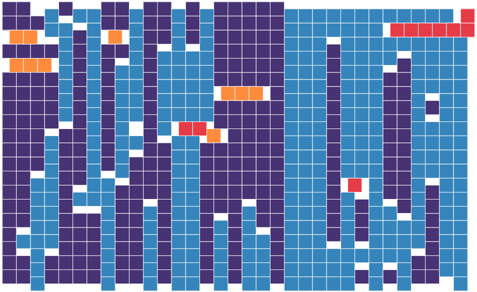
Joint work with Ron Peled



HUJI Dynamics Seminar,
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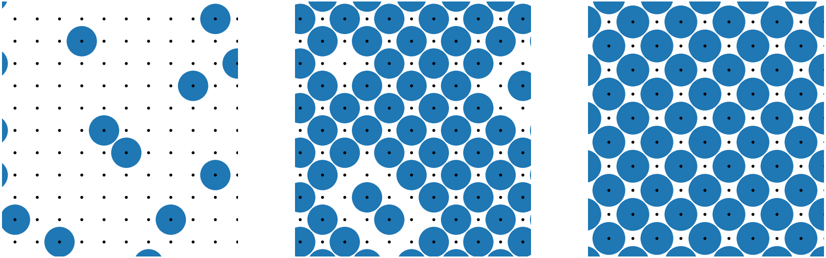
The hard-core lattice gas

- Domain: $\Lambda = (\mathbb{Z}/L\mathbb{Z})^2$ ("a discrete $L \times L$ torus").
- Configuration: a set $\sigma \subset \Lambda$ with no two points at distance one.
- Fugacity parameter: $\lambda > 0$.
- Probability measure: $\mu_{\Lambda,\lambda}(\sigma) = \frac{\lambda^{\#\sigma}}{Z_{\Lambda,\lambda}}$ where:
  - $\#\sigma$ is the number of points in σ.
  - $Z_{\Lambda,\lambda}$ is a normalization constant (the partition function).



Phase transition in the hard-core model

- Restrict to even L. The long-range order is captured by two events:
- $E_0 = \{99\%$ of the points in σ have an even sum of coordinates$\}$.
- $E_1 = \{99\%$ of the points in σ have an odd sum of coordinates$\}$.

Theorem (Dobrushin, 1968)

For all sufficiently large λ,

$$\lim_{\substack{L\to\infty \\ L \text{ even}}} \mu_{\Lambda,\lambda}(E_0 \cup E_1) = 1$$

- Dobrushin uniqueness condition: model is disordered at small λ.
- Open: Is there a single transition point λ_c from a disordered to an ordered state?

Continuum hard-core models

- Configuration σ consists of spheres in a domain in $\mathbb{R}^d$.
- Sampled with probability proportional to $\lambda^{\#\sigma}$ (with respect to a suitable Lebesgue measure).
- Major open problems:
  Is there an ordered state in dimensions $d \geq 3$?
  Is the rotational symmetry broken in dimension $d = 2$?
  Richthammer (2007): No translational-symmetry breaking in two dimensions.

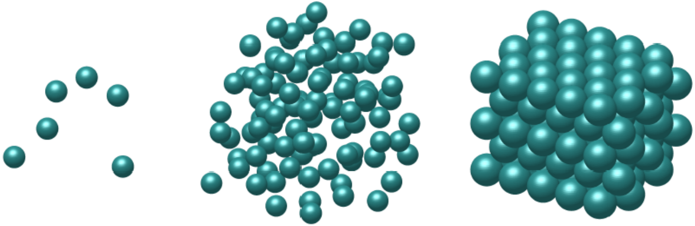


image credit: Ian Jauslin

Lattice hard-core models

- Comprehensive study by Mazel–Stuhl–Suhov (2018-19) of hard-core models on $\mathbb{Z}^2$, triangular and hexagonal lattices with general radius of exclusion.
- Prove long-range order at high fugacities in non-sliding cases.
- Sliding phenomenon:
  - Significant non-uniqueness of maximal density packings due to a sliding degree of freedom.
  - Occurs for a finite number of exclusion radii.
  - Unclear whether these cases still undergo a phase transition.

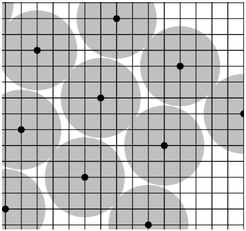
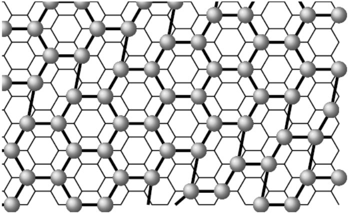
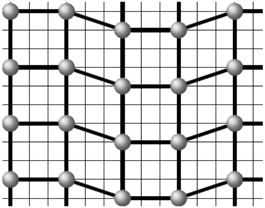


image credit: Izabella Stuhl

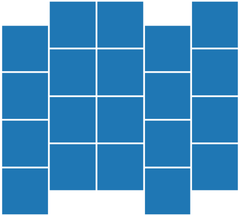
Lattice hard-core models: Sliding





Two cases where sliding occurs: $\mathbb{H}_2, D = 2 \quad \mathbb{Z}^2, D = 3$

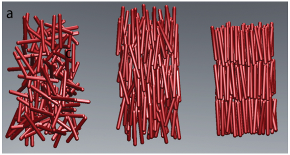
image credit: Mazel–Stuhl–Suhov



Sliding in the 2×2-hard-square model $(\mathbb{Z}^2, D = 2)$

Hard rod models

- Monomer-Dimer model:
  - Configuration consists of 2×1 rods (i.e., a matching).
  - Heilmann–Lieb (1972) famously proved the absence of a phase transition on all graphs.
- Many other models:
  - Onsager (isotropic-nematic transition in liquid crystals, 1949)
  - Heilmann–Lieb (1979) and Jauslin–Lieb (interacting monomer-dimer 2018).
  - Ioffe-Velenik-Zahradník (variable length rods on $\mathbb{Z}^2$, 2005)
  - Disertori–Giuliani (long rods on $\mathbb{Z}^2$, 2013),
  - Disertori–Giuliani–Jauslin (anisotropic plates in $\mathbb{R}^3$, 2020)



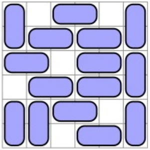
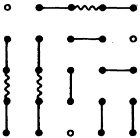
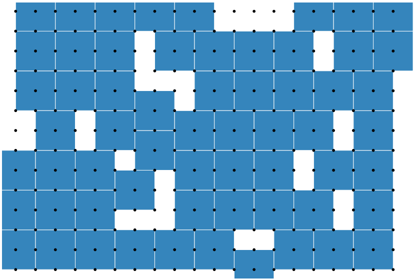



image credits: Zvonimir Dogic (2016), Nicolas Allegra (2015), Heilmann-Lieb (1979)

The 2×2 hard squares model

- Domain: $\Lambda = (\mathbb{Z}/L\mathbb{Z})^2$ ("a discrete $L \times L$ torus").
- Configuration: a set σ of pairwise disjoint 2×2 tiles with centers in Λ.
- Fugacity parameter: $\lambda > 0$.
- Probability of a configuration: $\mu_{\Lambda,\lambda}(\sigma) = \frac{\lambda^{\#\sigma - \frac{L^2}{4}}}{Z_{\Lambda,\lambda}}$ where:
  - $\#\sigma$ is the number of tiles in σ.
    ($-4\left(\#\sigma - \frac{L^2}{4}\right)$ counts vacant 1×1 squares).
  - $Z_{\Lambda,\lambda}$ is a normalization constant (the partition function).



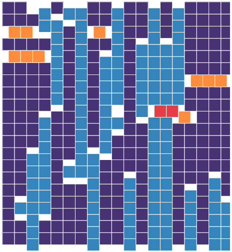
Main result: Columnar order

- Restrict to even L.
- Each tile has one of four parities:
  (Even,Even), (Even,Odd), (Odd,Even), (Odd, Odd).
- Let $E_{|,0}$, be the "ordering by even columns" event:
  more than 49% of the tiles have parity (Even, Even), and
  more than 49% of the tiles have parity (Even, Odd).
- Similarly define $E_{|,1}, E_{-,0}, E_{-,1}$.

Theorem (H.–Peled, 2021+)

For all sufficiently large λ,

$$\lim_{\substack{L\to\infty \\ L \text{ even}}} \mu_{\Lambda,\lambda}(E_{|,0} \cup E_{|,1} \cup E_{-,0} \cup E_{-,1}) = 1$$



an illustration of $E_{|,0}$

Properties of the ordered states

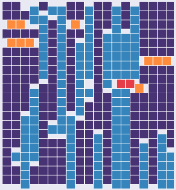
Theorem (H.–Peled, 2021+)

For sufficiently large λ, the following holds:

The set of doubly-periodic infinite volume Gibbs measures, is a simplex with four vertices (denoted $\mu_{\mathrm{ver},0}$, $\mu_{\mathrm{ver},1}$, $\mu_{\mathrm{hor},0}$ and $\mu_{\mathrm{hor},1}$).

These four measures are related to each other by translations and rotations.

One of them ($\mu_{\mathrm{ver},0}$) satisfies the following:



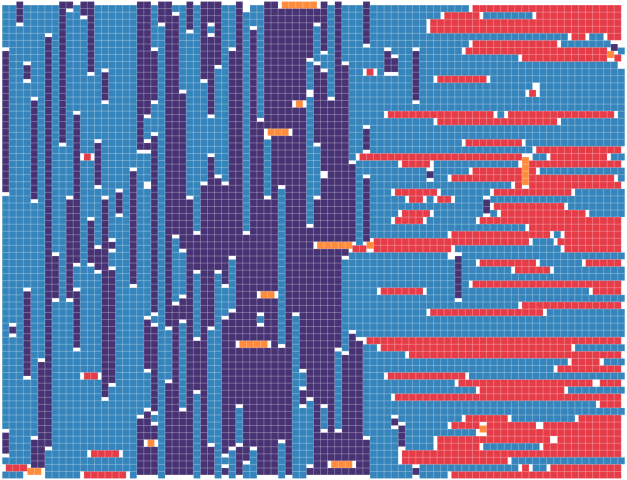
1. *$\mu_{\mathrm{ver},0}$ is $(2\mathbb{Z} \times \mathbb{Z})$-invariant and extremal.*
2. *Columnar order: $\mu_{\mathrm{ver},0}(\sigma(0,1)) = \Theta(\lambda^{-1})$.*
3. *Correlations decay exponentially with distance, for a non-isotropic distance function:*

$$d_{\mathrm{ver}}((x_1, y_1), (x_2, y_2)) := \lambda^{-1/2}|y_2 - y_1| + |x_2 - x_1|$$

Proof Ideas

- Note: we only discuss the proof of orientational order.

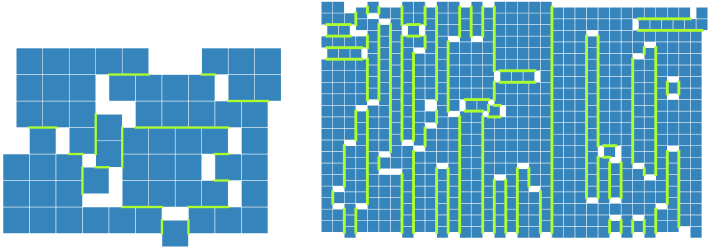
Interfaces between phases



Sticks

For a configuration σ, define:

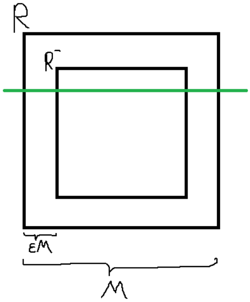
- a stick edge: a segment of length 1, bounding on tiles of different parities.
- a stick: a maximal path of stick edges.



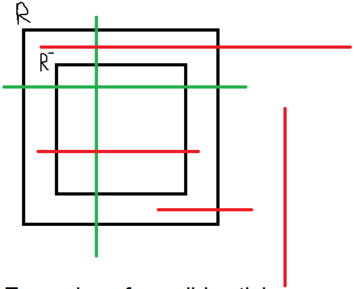
- Sticks cannot intersect. Thus two close long sticks must have same orientation.
- Bound the probability that most sticks are short, by direct calculation.

Properly divided squares (1/3)

- Let ϵ be a small constant.
- Set $M = M(\lambda)$ (think $M = \epsilon\lambda^{1/2}$)
- Let R be a $M \times M$ square
- define R^- as $(1-2\epsilon)M \times (1-2\epsilon)M$ square concentric to R. (assume $\epsilon M \in \mathbb{Z}$)
- Say R is properly divided (for σ) if a stick divides both R and R^-.

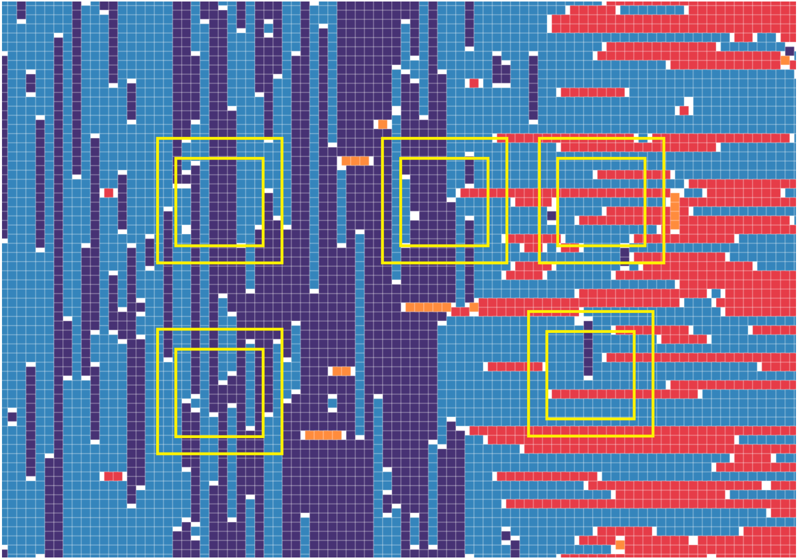


A properly dividing stick



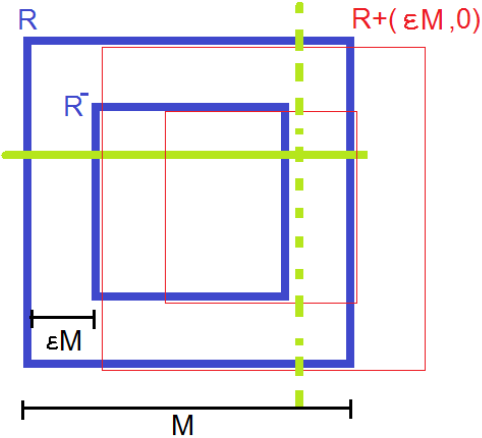
Examples of possible sticks. Green ones properly divide R and red ones don't.

Properly divided squares (2/3)



Properly divided squares (3/3)

- If R and $R + (\epsilon M, 0)$ are properly divided, they are properly divided in same orientation.
- Same for R and $R + (0, \epsilon M)$.



The main lemma

Let R be $M \times M$ for $M = \epsilon\lambda^{1/2}$. Denote by E_R the event that R is **not** properly divided.

Lemma

There is $\epsilon > 0$ such that for all sufficiently large λ,

$$\mu(E_R) \leq e^{-\epsilon^3 \lambda^{1/2}}$$

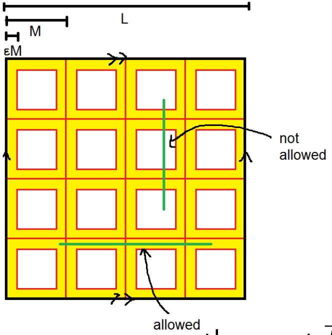
- In fact a multiplicative bound holds. If A is a set of copies of R shifted by vectors in $(M\mathbb{Z})^2$ then

$$\mu\Big(\bigcap_{R' \in A} E_R\Big) \leq e^{-\epsilon^3 \lambda^{1/2} |A|}$$

- This allows to prove orientational order with a Peierls argument.

The disseminated event

- Define the disseminated version of E_R to be $\overline{E_R} := \bigcap_{v \in (M\mathbb{Z})^2/(L\mathbb{Z})^2} E_{R+v}$.

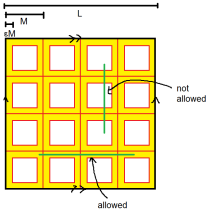


the event $\overline{E_R}$

- The 2×2 hard-square model is satisfies reflection positivity.
- Thus the chessboard estimate holds, and implies:

$$\mu(E_R) \leq \left(\mu(\overline{E_R})\right)^{\frac{M^2}{L^2}}$$

Bounding the disseminated event



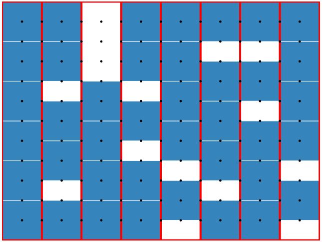
- In E_R all sticks have length $2M$ at most, except for sticks contained in the yellow regions.
- For simplicity we will discuss bounding the sum over the event E_M that all sticks have length at most $2M$.

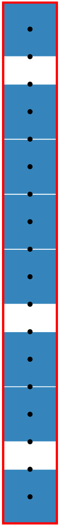
$$\mu(E_M) = \frac{\sum_{\sigma \in E_M} \lambda^{\#\sigma - \frac{L^2}{4}}}{Z_{\Lambda,\lambda}} \leq ?$$

- Bound separately the nominator and denominator.

Lower bound on $Z_{\Lambda,\lambda}$

- Consider a one dimensional system: $\Lambda = \{0\} \times \mathbb{Z}/L\mathbb{Z}$ for even L.
- Geometrically, configurations are packings in a $2 \times L$ rectangle.
- Easy to see that: $Z_{\{0\}\times\{1,2\dots,L-1\},\lambda} \geq \left(1+\lambda^{-1/2}\right)^{L/2}$
- Conclude for the torus:
  $Z_{(\mathbb{Z}/L\mathbb{Z})^2,\lambda} \geq (1+\lambda^{-1/2})^{L^2/4} \approx e^{\frac{1}{4}L^2\lambda^{-1/2}}$





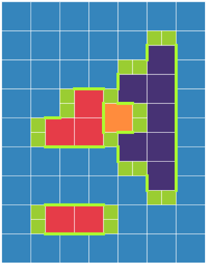
Bounding $\sum_{\sigma \in E_M} \lambda^{\#\sigma - L^2/4}$

- $\mathcal{H}_M = \{$components of vacancies and sticks that may appear in $\sigma \in E_M$, up to translation$\}$
- weight of a component: $w(c) := \lambda^{-\frac{1}{4}v(c)}$ where $v(c)$ is the number of vacancies in $c \in \mathcal{H}_M$.
- Next slide: $\sum_{c \in \mathcal{H}_M} w(c) = C\epsilon\lambda^{-1/2}$

  - For each component pick an arbitrary root.

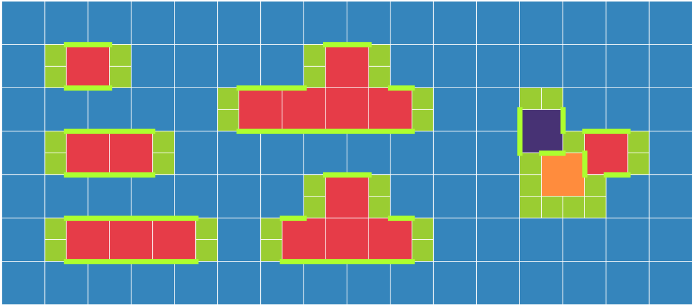
$$\sum_{\sigma \in E_M} \lambda^{\#\sigma - \frac{L^2}{4}} \leq \sum_{\sigma \in E_M} \prod_{v \in \Lambda^2} \begin{cases} w(c) & v \text{ root of } c \\ 1 & \text{o/w} \end{cases}$$

$$\leq \left(1 + \sum_{c \in \mathcal{H}_M} w(c)\right)^{L^2} \leq e^{C\epsilon\lambda^{-1/2}L^2}$$



Bounding $\sum_{c \in \mathcal{H}_M} w(c)$

- Proof idea: Sum $\lambda^{-\frac{1}{4}v}(2M)^d = (2\epsilon)^d \cdot \lambda^{(2d-v)/4}$ over "components up to the length of sticks" where d counts "degrees of freedom".



v=4 d=1 $2\epsilon\lambda^{-1/2}$

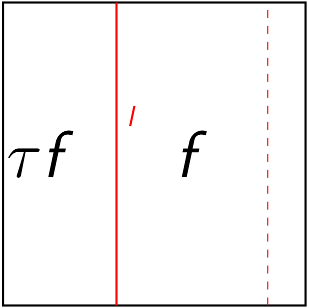
v=8 d=3 $(2\epsilon)^3\lambda^{-1/2}$

v=12 d=2 $(2\epsilon)^2\lambda^{-2}$

Reflection positivity

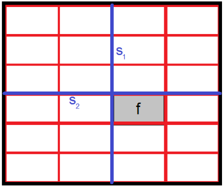
- l a vertical line through vertices of Λ
- l and its opposite divide Λ to two rectangles R_0, R_1.
- τ is the reflection through l
- τ exchanges R_0 with R_1.
- let f be R_0-local function.
- Conditioned on the restriction to l and its opposite, $\mu(f) = \mu(\tau f)$ and $\mu(f \cdot \tau f) = \mu(f) \cdot \mu(\tau f)$ thus, Reflection positivity: $\mu(f \cdot \tau f) \geq 0$

$(\mathbb{Z}/L\mathbb{Z})^2$



Reflection positivity

- for R_0-local f, g, define $< f, g >= \mu(f \cdot \tau g)$
- Reflection positivity: $< \cdot, \cdot >$ is a non-negative bilinear form.
- Thus the Cauchy-Schwarz inequality holds: $< f, g > \leq \sqrt{< f, f >< g, g >}$
- Example: $\mu(f) \leq \sqrt{\mu(f \cdot \tau_{s_1} f)} \leq \sqrt[4]{\mu(f \cdot \tau_{s_1} f \cdot \tau_{s_2} f \cdot \tau_{s_1} \tau_{s_2} f)}$



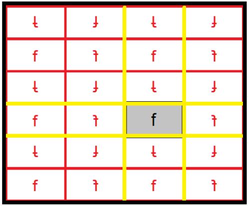
The chessboard estimate

- Let R be a rectangle.
- assume $2\mathrm{Width}(R), 2\mathrm{Height}(R)$ divide L.
- Let $T = T_\Lambda^R$ be the isometries generated by reflections in the sides of R.
- For each $\tau \in T$, let f_τ be R-local. (Then τf_τ is τR-local)
- Define a norm:

$$\mathfrak{z}_\Lambda^R(f) := \left[\mu \left(\prod_{\tau \in T} \tau f \right) \right]^{1/\# T}.$$

- Then

$$\mu \left(\prod_{\tau \in T} \tau f_\tau \right) \leq \prod_{\tau \in T} \mathfrak{z}_\Lambda^R(f_\tau)$$



Open problems

- A similar result is expected for $k \times k$ tiles, however, a different proof is needed since reflection positivity does not apply.
- What happens for $2 \times 2 \times 2$ cubes? We conjecture the existance of exactly 12 phases (of columnar order) at high fugacity.
- What happens for $1 \times k$ rods? At intermediate fugacity, a nematic phase was proved using cluster expansions (Disertori and Giuliani, 2013). What happens at high fugacity?

you for listening!