A minimal introduction to the Chessboard Estimate

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The Chessboard Estimate is an inequality, which holds for lattice systems in statistical mechanics if they satisfy a special property called Reflection Positivity. It is in a certain sense a generalization of the Cauchy-Schwarz inequality.

We will first define the Potts model, and state a theorem about it. Then we discuss Reflection Positivity, state the Chessboard Estimate and prove it. Finally, we show how it is used to prove the theorem about the Potts model.

This one-hour talk is self contained, and almost everything is given proof. This comes at the cost of giving a very narrow view of the subject. For introductions giving a wider view, see [1], [2, Chapter 10], [3, Section 2.7.1] and [4].

1 The Potts Model

For $d, L \in \mathbb{N}$, fix the domain to be a discrete torus: $\Lambda := (\mathbb{Z}/\mathbb{Z}L)^d = \{0, \dots, L-1\}^d$. Define a graph structure on Λ: for $u, v \in \Lambda$ say that $uv \in E(\Lambda)$ iff $u - v = \pm e_i$ for some i (with e_i denoting an element of the standard basis).

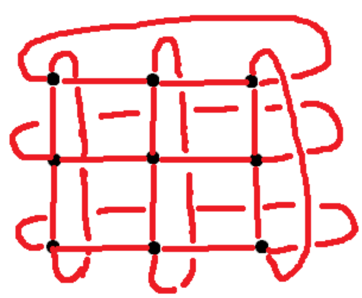

Figure 1: Λ for $d = 2$, $L = 3$

Configurations assign a spin from the set $\mathcal{S} := \{1, \dots, s\}$ to each site of Λ. The set of configurations is $\Omega = \{\sigma : \Lambda \to \mathcal{S}\}$.

For a configuration σ, an edge uv is called bad if $\sigma(u) \neq \sigma(v)$.

We now define a measure on Ω as follows.
Weight of configuration: $w(\sigma) = t^{\#\text{of bad edges in } \sigma}$.
Partition function: $Z = \sum_{\sigma \in \Omega} w(\sigma)$.
Probability measure: $\mathbb{P}(\sigma) = w(\sigma)/Z$.
To sum up, we remind that the model has four parameters: d, L, s and t.

Remark 1.1. The Ising model is the special case $s = 2$.

1.1 What we plan to prove:

Theorem 1.2 (long range order)**.** *For $d = 2$, and fixed s,*

$$\lim_{t \to 0} \sup_{L \in 2\mathbb{N}} \sup_{u,v \in \Lambda} \mathbb{P}(\sigma(u) \neq \sigma(v)) = 0$$

Remark 1.3. Note that for fixed L, $\lim_{t \to 0} \mathbb{P}(\sigma(u) \neq \sigma(v)) = 0$ is trivial since the probability that all spins are equal approaches 1.

Remark 1.4. Taking $d = 2$ is only for simplicity of presentation — the result holds for any $d \geq 2$. The limitation that L is even is not necessary, and is just an artifact of our proof technique. We will use "reflection positivity through vertices". One can overcome the limitation by using reflection positivity both through vertices and through edges.

Lemma 1.5 (multiplicative bound for bad edges)**.** *Fix $d = 2$. Let $L \in 2\mathbb{N}$. For each set $A \subset E(\Lambda)$,*

$$\mathbb{P}(\textit{all } e \in A \textit{ are bad}) \leq (c(s,t))^{|A|}$$

where $\lim_{t \to 0} c(s,t) = 0$.

We now prove the theorem using the lemma. In Section 4 we will deduce the lemma from the chessboard estimate.

Proof of Theorem. Denote by B the set of bad edges, and by B^* the set of their dual edges. Let $\mathcal{C}$ be a set of cycles in the dual graph of Λ:

$$\mathcal{C} := \left\{ \substack{\text{contractible cycles} \\ \text{around } u \text{ or } v} \right\} \cup \left\{ \substack{\text{non-contractible} \\ \text{cycles}} \right\}$$

Then whenever $\sigma(u) \neq \sigma(v)$, there is $A^* \in \mathcal{C}$ for which $A \subset B$ (where A is the set of dual edges of A^*).

For $u = (x, y)$, a contractible cycle $A^* \in \mathcal{C}$ around u in the dual graph can be specified as a sequence $(w_0, \ldots, w_{|B|})$ with $w_0 = (x + a + 1/2, y + 1/2)$ with $0 < a < |A|$ and with $w_{i+1} - w_i \in \{(1,0), (-1,0), (0,1), (0,-1)\}$ with consecutive increments never summing to 0. Taking into account cycles around v and non-contractible cycles, we have

$$|\{A^* \in \mathcal{C} \mid |A| = k\}| \leq 4k3^k.$$

By a union bound

$$\mathbb{P}(\sigma(u) \neq \sigma(v)) \leq \sum_{C^* \in \mathcal{C}} \mathbb{P}(\text{all } e \in A \text{ are bad})$$
$$\leq \sum_{k=4} 4k3^k \left(c(s,t)\right)^k \xrightarrow{t \to 0} 0$$

□

2 Reflection Positivity

Assume that L is even. Consider two "halves" of the torus:

$$\Lambda_0 = \{0, \ldots, L/2\} \times \{0, \ldots, L-1\}^{d-1}$$
$$\Lambda_1 = \{-L/2, \ldots, 0\} \times \{0, \ldots, L-1\}^{d-1}$$

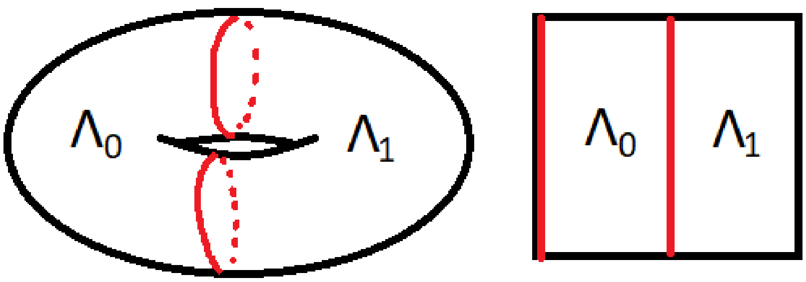

Figure 2: Two "halves" of Λ. In red appears $\Lambda_0 \cap \Lambda_1$.

And note that $\Lambda_0 \cap \Lambda_1 = \{0, L/2\} \times \{0, \ldots, L-1\}^{d-1}$. Let τ_0 be the reflection through the intersection of the halves: $\tau_0(x,y) = (-x,y)$ (what we do here applies for general dimension, but we fall back to $d = 2$ whenever this simplifies notation). Note that $\tau_0(\Lambda_0) = \Lambda_1$ and $\tau_0(\Lambda_1) = \Lambda_0$.

Consider the vector space of functions $V_{\Lambda_0} := \{f : \mathcal{S}^{\Lambda_0} \to \mathbb{R}\}$ (think of its elements as RVs that depend only on half of the configuration — call them Λ_0-local functions).

For $f \in V_{\Lambda_0}$ and $\sigma : \Lambda \to \mathcal{S}$ we will abuse notation by writing $f(\sigma)$ instead of the more accurate $f(\sigma_{|\Lambda_0})$.

For $f : \mathcal{S}^\Lambda \to \mathbb{R}$ define τf by $\tau f(\sigma) = f(\sigma \circ \tau)$.

Define a bilinear form on V_{Λ_0}

Definition 2.1 (reflection product)**.**

$$< f, g >:= \mathbb{E}[f \cdot \tau_0 g]$$

Note that $\tau_0 g$ is Λ_1-local.

Claim 2.2. $< f, g >=< g, f >$

Proof. $< f, g >= \mathbb{E}[f \cdot \tau_0 g] = \mathbb{E}[\tau_0(g \cdot \tau_0 f)] = \mathbb{E}[g \cdot \tau_0 f] =< g, f >$ □

Claim 2.3 (Reflection Positivity). The bilinear form $< \cdot, \cdot >$ is non-negative.

Proof. Let $\omega : \Lambda_0 \cap \Lambda_1 \to \mathcal{S}$ and define $E = \{\sigma \in \Omega \,|\, \sigma_{|\Lambda_0 \cap \Lambda_1} = \omega\}$

$$\mathbb{E}[f \cdot \tau f \,|\, E] \overset{(*)}{=} \mathbb{E}[f|\, E] \cdot \mathbb{E}[\tau f \,|\, E] = (\mathbb{E}[f|\, E])^2 \geq 0.$$

□

Exercise 2.4. Justify $(*)$. (This is a special case of the domain Markov property).

Corollary 2.5 (Cauchy-Schwarz)**.**

$$< f, g > \leq \sqrt{< f, f >< g, g >}$$

Proof. WLOG $< f, f >=< g, g >= 1$, by homogeneity. Expand $0 \leq < f - g, f - g >$. □

Example 2.6. Let E be an event defined in terms of $\sigma_{|\Lambda_0}$ (a Λ_0-local event). Then E is positively correlated with τE.

Proof.

$$\begin{aligned} \mathbb{P}(\tau E) = \mathbb{P}(E) &= \mathbb{E}[\mathbf{1}_E \cdot \tau 1] =< \mathbf{1}_E, 1 > \\ &\leq \sqrt{< \mathbf{1}_E, \mathbf{1}_E > \cdot < 1, 1 >} \\ &= \sqrt{\mathbb{E}[\mathbf{1}_E \cdot \tau \mathbf{1}_E] \cdot \mathbb{E}[1 \cdot \tau 1]} = \sqrt{\mathbb{P}(E \cap \tau E)} \end{aligned}$$

thus $\mathbb{P}(E \cap \tau E) \geq \mathbb{P}(E) \cdot \mathbb{P}(\tau E)$ □

Example 2.7. Let $d = 2$. For each $1 \leq k \leq L$ define a set of edges $A_k = \{(x, y)(x+1, y) \,|\, 0 \leq x < k\}$. Define E_k to be the event that all $e \in A_k$ are bad. Define τ_k to be the reflection through $\{x = k\}$, $\tau_k(x, y) = (2k - x, y)$. Then $E_{2k} = E_k \cap \tau_k E_k$. By the previous example applied with τ_k instead of τ_0, we have

$$\mathbb{P}(E_k) = \mathbb{P}(\tau_k E_k) \leq \sqrt{\mathbb{P}(E_{2k})}.$$

Assume that $L = 2^n$. Then applying the above repeatedly gives $\mathbb{P}(E_1) \leq \sqrt[L]{\mathbb{P}(E_L)}$.

$$\mathbb{P}\left(E_L\right) \leq \frac{1}{Z} \sum_{\sigma \in E_L} w(\sigma) \leq |\mathcal{S}|^{L^2} \max_{\sigma \in E_L} w(\sigma) \leq |\mathcal{S}|^{L^2} \cdot t^{L^2}$$

since for a configuration in E_L there are L^2 bad edges, $|E_L| \leq |\mathcal{S}|^{L^2}$ and $Z \geq 1$. This gives $\mathbb{P}(E_1) \leq (st)^L$. Note that this is a special case of Lemma 1.5, with $c(s, t) = st$ and $A = A_1$.

3 The Chessboard estimate

Fix M such that $2M$ divides L. Denote $R = \{0, \dots, M\}^d \subset \Lambda$. Consider the vector space $V_R = \{f : \mathcal{S}^R \to \mathbb{R}\}$ (the space of R-local functions). Define a set of torus isometries (isomorphisms) G. Define G to be the group **generated** by all reflections τ_l where l is of the form $\{x = C\}$ or $\{y = C\}$ and M divides C (where τ_l denoted a reflection through the hyperplane l).

Claim 3.1. There is a 1-1 correspondence between squares $R' = R + (xM, yM)$ and elements τ of G, where $\tau R = R'$. In particular, $|G| = \left(\frac{L}{M}\right)^d$.

Proof. It is clear that there is $\tau \in G$ for each R'. Reflections through orthogonal hyperplanes commute, thus it suffices to consider $d = 1$. It remains to see that $G = L/M$. The composition of two reflections is a translation by a multiple of $2M$. Since $2M$ divides L, translations by odd multiples of M are not in G, so G has $L/2M$ translations, and $L/2M$ reflections. □

Definition 3.2 (Chessboard product)**.** For $(f_\tau)_{\tau \in G} \in (V_R)^G$ define

$$< f_\tau >_{\tau \in G} := \mathbb{E}[\prod_{\tau \in G} \tau f_\tau]$$

Definition 3.3 (The Chessboard "norm")**.** For $f \in V_R$ define

$$\|f\|_R := \sqrt[(L/M)^d]{< f >_{\tau \in G}} = \sqrt[(L/M)^d]{\mathbb{E}[\prod_{\tau \in G} \tau f]}$$

$\|\cdot\|_R$ is obviously **homogeneous**. The expectation inside the root is indeed nonnegative, however it may be zero for non-zero f. Thus it would be more proper to call it a seminorm rather than a norm. Finally, to show that it is a seminorm, we must prove subaditivity. This we do not prove now, and we do not assume it.

Exercise 3.4. Show that for sufficiently large M, there is $0 \neq f \in V_R$ with $\|f\|_R = 0$.

To simplify notation, consider the case $d = 1$,$L/M = 6$. Recall that $R = \{0, \dots, M\}$ and define $\tau_0, \dots, \tau_5$ by $\tau_i R = R + iM$. We now denote tuples indexed by G as $(f_{\tau_0}, f_{\tau_1}, f_{\tau_2}, f_{\tau_3}, f_{\tau_4}, f_{\tau_5}) := (f_\tau)_{\tau \in G}$. Then $< a, b, c, d, e, f >= \mathbb{E}[a \cdot \tau_1 b \cdot \tau_2 c \cdot \tau_3 d \cdot \tau_4 e \cdot \tau_5 f]$. Denote $h_0 = \tau_0 a \cdot \tau_1 b \cdot \tau_2 c$ and $h_1 = \tau_0 f \cdot \tau_1 e \cdot \tau_2 d$. Note that $h_0, h_1 \in V_{\Lambda_0}$ and $< a, b, c, d, e, f >=< h_0, h_1 >$. If $a = b = c = d = e = f$, then $h_0 = h_1$ and by the non-negativity of the reflection product we have $< f, f, f, f, f, f >=< h_0, h_0 >\geq 0$, showing that the Chessboard norm is well defined.

Claim 3.5. For $a, b, c, d, e, f \in V_R$,

$$< a, b, c, d, e, f >\leq \sqrt{< a, b, c, c, b, a >< f, e, d, d, e, f >}$$

Proof. By Cauchy-Schwarz, LHS $=< h_0, h_1 > \leq \sqrt{< h_0, h_0 >< h_1, h_1 >}$ = RHS. $\square$

The claim holds analogously for other reflections. For example, for τ_M we get $< a, b, c, d, e, f > \leq \sqrt{< a, a, f, e, e, f >< b, b, c, d, d, c >}$.

Theorem 3.6 (The Chessboard estimate)**.** *Let* $(f_\tau)_{\tau \in G} \in (V_R)^G$ *then*

$$< f_\tau >_{\tau \in G} \leq \prod_{\tau \in G} \|f_\tau\|_R$$

Proof of Theorem 3.6. Excuse #1: $\{f \in V_R : \|f\|_R \neq 0\}$ is dense (and open) in V_R, thus by continuity we may assume $\|f_\tau\|_R \neq 0$ for each $\tau \in G$.
Excuse #2: By scaling each of the f_τ, using homogeneity we may assume $\|f_\tau\|_M = 1$ for each $\tau \in A$.
Excuse #3: We prove for the case $d = 1$ and $L/M = 6$, since this suffices to convey the idea.

Denote $F = \{f_\tau : \tau \in G\}$.

Thus it suffices to prove that

$$M := \max_{g \in F^G} < g > \leq 1$$

Take some $g_0 \in F^G$ with $< g_0 >= M$. Denote $g_0 = (a, b, c, d, e, f)$. Denote $g_1 = (a, b, c, c, b, a), g_1' = (f, e, d, d, e, f)$. Then $\max\{< g_1 >, < g_1' >\} \leq M$ by the definition of M. But $M =< g_0 > \leq \sqrt{< g_1 >< g_1' >} \leq \sqrt{M \cdot M}$. Thus $< g_1 >= M$.

Similarly we define $g_2 = (a, a, a, b, b, a), g_2' = (b, b, c, c, c, c)$ and show that $< g_2 >= M$. Again similarly we show $< g_3 >= M$ for $g_3 = (a, a, a, a, a, a)$. But then $M = \|a\|_R^6 = 1$. $\square$

Exercise 3.7. Justify Excuse #1.

Exercise 3.8. Prove that for $f, g \in V_R$, $\|f + g\|_R \leq \|f\|_R + \|g\|_R$. Hint: think how Cauchy-Schwarz implies the triangle inequality.

4 Application

Proof of Lemma 1.5. Let $M = 1, R = \{0, 1\}^d$. Let E be the event that two sites in R have distinct spins.

$$\|E\|_R = \sqrt[L^2]{\mathbb{P}(\bigcap_{\tau \in G} \tau E)} = \sqrt[L^2]{\mathbb{P}(\bigcap_{v \in \Lambda} E + v)} \leq \sqrt[L^2]{s^{L^d} t^{L^d/2}} = s\sqrt{t}$$

Let $A \subset E(\Lambda)$

Set $B = \{\tau \in G \,|\, \exists e \in A, e \subset \tau R\}$. Note that $|B| \geq |A|/2$. Then

$$\mathbb{P}(\bigwedge_{uv \in A} \sigma(u) \neq \sigma(v)) \leq \mathbb{E}\left[\prod_{\tau \in G} \begin{cases} \mathbf{1}_{\tau E} & \tau \in B \\ 1 & \tau \notin B \end{cases}\right]$$

$$\leq \prod_{\tau \in G} \begin{cases} \|E\|_R & \tau \in B \\ 1 & \tau \notin B \end{cases} = \|E\|_R^{|B|} \leq \left(s\sqrt{t}\right)^{|A|/2}$$

which suffices for $c(s,t) = \sqrt{s\sqrt{t}}$. □

References

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